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Laminar incompressible flow in the entrance region of a square duct.

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IN THE ENTRANCE REGION OF A SQUARE SUCT

by

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UNITED STATES
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THESIS

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[redacted]

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ABSTRACT

The development of the three dimensional, laminar velocity profile in the entrance length of a rectangular duct is investigated. The solution to this hydrodynamic problem is obtained from the full, incompressible Navier-Stokes equations and the continuity equation, in finite difference form, on the digital computer employing the computational method of Chorin (1). The solution yields the hydrodynamic velocities U, V, and W and the friction factor as a function of the distance from the entrance.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

TABLE OF SYMBOLS

English Letter

A	Cross sectional area of the duct
d	Hydraulic diameter, $\frac{4A}{D}$
D	Perimeter of the duct
F	Friction factor
P	Pressure
$\bar{P}$	Dimensionless pressure
t	Time
T	Dimensionless time
u_o	Axial reference velocity
u	Axial velocity
$\bar{u}$	Mean axial velocity
U	Dimensionless axial velocity, u/u_o
$\bar{U}$	Mean dimensionless axial velocity, $\bar{u}/u_o = 1$
v	Crosswise velocity in y direction
V	Dimensionless crosswise velocity in y direction
w	Crosswise velocity in z direction
W	Dimensionless crosswise velocity in z direction
x	Axial normal coordinate
X	Dimensionless length, $\frac{x}{d}$
X_e	Dimensionless entrance region length
y	Crosswise normal coordinate
y_w	Value of y at the sidewall boundary

Y	Dimensionless normal coordinate
z	Crosswise normal coordinate
z_w	Value of z at the sidewall boundary
Z	Dimensionless normal coordinate

Greek Letter

δ	Artificial compressibility
η	Crosswise coordinate, y or z
ρ	Fluid density, artificial density
τ	Shear stress
μ	Fluid viscosity
ν	Kinematic viscosity

Nondimensional Groups

$\frac{\rho u_0 d}{\mu}$	Reynolds number, R_e

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I. INTRODUCTION

The design of compact heat exchangers for use in gas turbine regenerators, among other applications, entails the maximum use of the high heat transfer rates attainable in the entrance region. In this region the fluid is undergoing rapid transition from its uniform profile at the entrance to its fully established value farther downstream. A knowledge of the temperature profiles in the region is required to obtain the heat transfer rates. The temperature profile develops simultaneously with the velocity profile and is dependent upon it. The Graetz solution assumes a fully established velocity profile and thus eliminates most of the complexity of the hydrodynamic problem. This solution, however, fails to yield accurate results in problems involving large entrance region to overall length ratios.

Various approximate methods have been devised for two dimensional channels, such as a circular pipe or parallel plate channels, to determine the development of velocity profiles. Until recently these solutions have invariably made boundary layer assumptions. Schlichting (5) obtained a solution for the flow between a pair of infinite parallel flat plates. He used two asymptotic series solutions, one based on Blasius' solution of the boundary layer development expanded in the downstream direction, and the other based on the Hagen-Poiseuille solution of a parabolic velocity distribution, expanded in the upstream direction. He then joined these two solutions.

Recently Wang and Longwell (6) have employed the full momentum equations and the introduction of a stream function to obtain a solution for the velocity development between parallel flat plates. The resulting equations were cast in finite difference form and the solution effected numerically. To date, no exact, three dimensional solutions for developing velocity profiles have been obtained.

The purpose of this analysis is to present an exact, finite-difference solution to the full incompressible Navier-Stokes equations for the hydrodynamic entrance region of a square duct. This solution is not nearly as simple as approximate methods. Its value, however, lies in its potential accuracy and the fact that it is an exact solution on which the credibility of various approximate solutions may be based.

II. ANALYSIS

A. Governing Equations

The governing equations are the momentum (Navier-Stokes) and the continuity equations. The complexity of the equations requires the following assumptions concerning the flow:

1. steady and three dimensional
2. laminar
3. incompressible

Through these assumptions, the energy and momentum equations become uncoupled, and may be expressed as follows:

Case I:

Momentum:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] \tag{3}$$

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

B. <u>Boundary Conditions</u>

The boundary conditions associated with the equations for a square duct with a uniform velocity at the entrance are:

1. At the entrance ($x = 0$)

   $v = w = 0$

   $u = u_o$

2. At the walls ($x = 0$, $y = y_w$, $z = 0$, $z = z_w$);

   $u = v = w = 0$

C. <u>Non-dimensionalization</u>

Equations (1), (2), (3) and (4) may be written in dimensionless form through the introduction of the following dimensionless variables:

$$U = \frac{u}{u_o} \quad V = \frac{v}{u_o} \quad W = \frac{w}{u_o} \quad \bar{P} = \frac{d}{\rho \nu u_o} \quad R_e = \frac{\rho u_o d}{\mu}$$
$$X = \frac{x}{d} \quad Y = \frac{y}{d} \quad Z = \frac{z}{d} \quad T = \frac{\nu \tau}{d^2} \tag{5}$$

The non-dimensional equations are then:

Momentum:

$$\frac{\partial U}{\partial T} + R_e\left[U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} + W\frac{\partial U}{\partial Z}\right] = -\frac{\partial \bar{P}}{\partial X} + \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} + \frac{\partial^2 U}{\partial Z^2} \tag{6}$$

$$\frac{\partial V}{\partial T} + R_e\left[U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} + W\frac{\partial V}{\partial Z}\right] = -\frac{\partial \bar{P}}{\partial Y} + \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} + \frac{\partial^2 V}{\partial Z^2} \tag{7}$$

$$\frac{\partial W}{\partial T} + R_e\left[U\frac{\partial W}{\partial X} + V\frac{\partial W}{\partial Y} + W\frac{\partial W}{\partial Z}\right] = -\frac{\partial \bar{P}}{\partial Z} + \frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} + \frac{\partial^2 W}{\partial Z^2} \tag{8}$$

Continuity:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{\partial W}{\partial Z} = 0 \tag{9}$$

D. Friction Factor

To obtain an equation for the friction factor, the relation between the shear stress and the flow variables must first be introduced:

$$\tau = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} \quad \text{or} \quad \tau = \mu \frac{\partial u}{\partial z}\bigg|_{z=0} \tag{10}$$

Since the duct is symmetrical, either derivative, $\frac{\partial u}{\partial y}$ or $\frac{\partial u}{\partial z}$, may be used in equation (10). By introducing a new variable η, where $\eta = y$ or z, equation (10) may be written:

$$\tau = \mu \frac{\partial u}{\partial \eta}\bigg|_{\eta=0} \tag{11}$$

The friction factor, F, may be related to the shear stress by the equation:

$$F = \frac{8\tau}{\rho \bar{u}^2} \tag{12}$$

By substituting equation (11) into (12) the desired equation for the friction factor is obtained:

$$F = \frac{8\mu}{\rho \bar{u}^2} \frac{\partial u}{\partial \eta}\bigg|_{\eta=0} \tag{13}$$

By defining a new dimensionless variable, $\bar{\eta} = \eta/d$, equation (13) may be written in dimensionless form

$$F = \frac{8}{R_e} \frac{\partial U}{\partial \bar{\eta}}\bigg|_{\bar{\eta}=0} \tag{14}$$

III. METHOD OF SOLUTION

A. Technique

The recently formulated technique of Chorin (1) was adopted for the solution of the governing equations. The principle of the method of solution suggested by Chorin lies in the introduction of an artificial compressibility, δ, into the equations of motion in such a way that the final results do not depend on δ. The solution is in essence a relaxation method. In employing this method, it is necessary to introduce a modified set of governing equations:

Momentum:

$$\frac{\partial U}{\partial T} + R_e\left[U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} + W\frac{\partial U}{\partial Z}\right] = -\frac{\partial \bar{P}}{\partial X} + \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} + \frac{\partial^2 U}{\partial Z^2} \tag{15}$$

$$\frac{\partial V}{\partial T} + R_e\left[U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} + W\frac{\partial V}{\partial Z}\right] = -\frac{\partial \bar{P}}{\partial Y} + \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} + \frac{\partial^2 V}{\partial Z^2} \tag{16}$$

$$\frac{\partial W}{\partial T} + R_e\left[U\frac{\partial W}{\partial X} + V\frac{\partial W}{\partial Y} + W\frac{\partial W}{\partial Z}\right] = -\frac{\partial \bar{P}}{\partial Z} + \frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} + \frac{\partial^2 W}{\partial Z^2} \tag{17}$$

Continuity:

$$\frac{\partial \rho}{\partial T} + \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{\partial W}{\partial Z} = 0 \tag{18}$$

State:

$$\bar{P} = \rho/\delta \tag{19}$$

In this set of auxiliary equations ρ is an artificial density, δ is an artificial compressibility, and $\bar{P} = \rho/\delta$ is an artificial equation of state.

Equations (15), (16), (17) and (18) may be put into finite difference form by introducing the following finite difference approximations:

$$\frac{\partial U}{\partial X} = \frac{U(X+\Delta X, Y, Z, T) - U(X-\Delta X, Y, Z, T)}{\Delta X} = \frac{U(X^+) - U(X^-)}{\Delta X}$$

$$\frac{\partial^2 U}{\partial X^2} = \frac{U(X+\Delta X, Y, Z, T) + U(X-\Delta X, Y, Z, T) - U(X, Y, Z, T+\Delta T) - U(X, Y, Z, T-\Delta T)}{\Delta X^2}$$

$$= \frac{U(X^+) + U(X^-) - U(T^+) - U(T^-)}{\Delta X^2}$$

The set of modified equations put into finite difference form and solved for the appropriate variables yields the following computing equations:

$$\begin{aligned} U(T^+) = \frac{1}{1 + \frac{2\Delta T}{\Delta X^2} + \frac{2\Delta T}{\Delta Y^2} + \frac{2\Delta T}{\Delta Z^2}} \Bigg\{ & -\frac{R_e \Delta T}{\Delta X}\left[U^2(X^+) - U^2(X^-)\right] \\ & - \frac{R_e \Delta T}{\Delta Y}\left[U(Y^+)V(Y^+) - U(Y^-)V(Y^-)\right] - \frac{R_e \Delta T}{\Delta Z}\left[U(Z^+)W(Z^+) - U(Z^-)W(Z^-)\right] \\ & + \frac{2\Delta T}{\Delta X^2}\left[U(X^+) + U(X^-) - U(T^-)\right] + \frac{2\Delta T}{\Delta Y^2}\left[U(Y^+) + U(Y^-) - U(T^-)\right] \\ & + \frac{2\Delta T}{\Delta Z^2}\left[U(Z^+) + U(Z^-) - U(T^-)\right] - \frac{\Delta T}{\delta \Delta X}\left[\rho(X^+) - \rho(X^-)\right] + U(T^-) \Bigg\} \end{aligned} \tag{20}$$

$$V(T^+) = \frac{1}{1 + \frac{2\Delta T}{\Delta X^2} + \frac{2\Delta T}{\Delta Y^2} + \frac{2\Delta T}{\Delta Z^2}} \Bigg\{ - \frac{R_e \Delta T}{\Delta X}\Big[U(X^+)V(X^+) - U(X^-)V(X^-)\Big]$$

$$- \frac{R_e \Delta T}{\Delta Y}\Big[V^2(Y^+) - V^2(Y^-)\Big] - \frac{R_e \Delta T}{\Delta Z}\Big[W(Z^+)V(Z^+) - W(Z^-)V(Z^-)\Big]$$

$$+ \frac{2\Delta T}{\Delta X^2}\Big[V(X^+) + V(X^-) - V(T^-)\Big] + \frac{2\Delta T}{\Delta Y^2}\Big[V(Y^+) + V(Y^-) - V(T^-)\Big]$$

$$+ \frac{2\Delta T}{\Delta Z^2}\Big[V(Z^+) + V(Z^-) - V(T^-)\Big] - \frac{\Delta T}{\delta \Delta Y}\Big[\rho(Y^+) - \rho(Y^-)\Big] + V(T^-)\Bigg\} \tag{21}$$

$$W(T^+) = \frac{1}{1 + \frac{2\Delta T}{\Delta X^2} + \frac{2\Delta T}{\Delta Y^2} + \frac{2\Delta T}{\Delta Z^2}} \Bigg\{ - \frac{R_e \Delta T}{\Delta X}\Big[U(X^+)W(X^+) - U(X^-)W(X^-)\Big]$$

$$- \frac{R_e \Delta T}{\Delta Y}\Big[V(Y^+)W(Y^+) - V(Y^-)W(Y^-)\Big] - \frac{R_e \Delta T}{\Delta Z}\Big[W^2(Z^+) - W^2(Z^-)\Big]$$

$$+ \frac{2\Delta T}{\Delta X^2}\Big[W(X^+) + W(X^-) - W(T^-)\Big] + \frac{2\Delta T}{\Delta Y^2}\Big[W(Y^+) + W(Y^-) - W(T^-)\Big]$$

$$+ \frac{2\Delta T}{\Delta Z^2}\Big[W(Z^+) + W(Z^-) - W(T^-)\Big] - \frac{\Delta T}{\delta \Delta Z}\Big[\rho(Z^+) - \rho(Z^-)\Big] + W(T^-)\Bigg\} \tag{22}$$

$$\rho(T^+) = -\frac{\Delta T}{\Delta X}\Big[U(X^+) - U(X^-)\Big] - \frac{\Delta T}{\Delta Y}\Big[V(Y^+) - V(Y^-)\Big] - \frac{\Delta T}{\Delta Z}\Big[W(Z^+) - W(Z^-)\Big] + \rho(T^-) \tag{23}$$

B. Boundary Conditions

In dimensionless form the boundary conditions for equations (15), (16), (17); and (18) are:

1. At the entrance: (X = 0):

$V = W = 0$

$U = 1$

2. At the walls (Y = 0, 1 Z = 0, 1):

$W = V = U = 0$

3. At X = X entrance length:

$W = V = 0$

$$U = U_{\text{fully established}}$$

$$= \frac{2 F R_e}{\pi^3} \sum_{N=1,3,5,\ldots}^{\infty} \frac{1}{N^3} (-1)^{\frac{N-1}{2}} \left[1 - \frac{\cosh(N\pi Y)}{\cosh(N\pi/2)} \right] \cos(N\pi Z)$$

C. Stability

With the introduction of the modified equations (15), (16), (17), (18), and (19) an artificial Mach number is also introduced.

$$M = R_e \delta^{1/2} \max_{(\text{FIELD})} \left(U^2 + V^2 + W^2 \right)^{1/2}$$

Chorin (1) has stated that, in order to insure stability, the flow Mach number must be kept less than one. An additional requirement that must be met for stability of the set is that:

$$\Delta T \leq .35682 \min_{(\text{FIELD})} \left(\Delta X \text{ or } \Delta Y \text{ or } \Delta Z \right) \delta^{1/2}$$

D. Calculation of F

In order to calculate the friction factor, it is necessary to introduce a forward difference approximation as follows:

$$\frac{\partial U}{\partial \bar{\eta}} = \frac{U(\bar{\eta}+\Delta\bar{\eta}) - U(\bar{\eta})}{\Delta\bar{\eta}} = \frac{U(\bar{\eta}^{+}) - U(\bar{\eta})}{\Delta\bar{\eta}}$$

Using this approximation, equation (14) may be written:

$$F = \frac{8}{R_e} \left. \frac{U(\bar{\eta}^{+})}{\Delta\bar{\eta}} \right|_{\bar{\eta}=0} \tag{24}$$

Since equation (14) is applied at $\bar{\eta} = 0$, $U(\bar{\eta})$ is identically zero and is eliminated from the finite difference expression.

Equation (24) may be used to obtain an average value of the friction factor. An average value of $\left.\frac{\partial U}{\partial \bar{\eta}}\right|_{\bar{\eta}=0}$ may be determined at each X-wise location by summing $U(\bar{\eta}^{+})$ at each grid point on the boundary and dividing by the number of points used in the summation. The average value of the slope of the velocity profile may now be used to calculate an average friction factor as follows:

$$F = \frac{8}{N \, \Delta\bar{\eta} \, R_e} \sum U_{w+1} \tag{25}$$

N = number of points used

U_{w+1} = velocity at one grid point from the side wall boundary

The grid system and summation points are shown in Figure 1.

E. Numerical Procedure

A finite difference solution was obtained, using the IBM system 360 digital computer. Equations (20), (21), (22), and (23) involve approximating

a solution for the entire flow field at time equal to T and T^- and then calculating a new approximation for time equal to T^+. The computed values of U, V, and W are tested for continuity in equation (23), and if $\rho(T^+) - \rho(T^-)$ does not equal zero, then a steady state solution has not yet been achieved. T is then set equal to T^+ and T^- equal to T. Then the process is repeated until a steady solution is reached. That is until $\partial\rho/\partial T = 0$. This sequence serves to relax the set of equations.

Solution also requires that $\partial\rho/\partial T$ be approximated at the grid points on the side wall boundary since equation (23) cannot be used at these points. To apply equation (23), one would have to impose the condition that the flow parameters are identical on either side of the channel walls. This assumption, however, is unrealistic. Instead, the following approximation is introduced at the side wall boundary:

$$\rho(T^+) = -\frac{3\Delta T}{\Delta Y} V(Y^+) + \rho(T^-)$$

Using the stability requirements, the solution to equations (20), (21), (22), and (23) was obtained using the following values of grid parameters:

$$\Delta X = .00164 \qquad \Delta T = .000001 \qquad \Delta Y = \Delta Z = .1 \qquad \zeta = .16$$

The value of ΔX was obtained using the entrance length of .0328, computed by McComas (4), for a hydraulic diameter of 1.0 and a Reynolds number of 1.0. The fully established velocity profile at this entrance length was taken from Knudsen (2). In this series solution for the fully established value of U, the pressure gradient $\partial\bar{P}/\partial X$ was replaced by $\rho\bar{U}^2 F/2d$, where F is the fully established factor, also taken from Knudsen (2).

The value of F computed by Knudsen (56.24) agrees well with the value suggested by Lundgren (3) (56.908) and McComas (4) (56.908).

The computer program written in Fortran IV is included in the appendix.

III. RESULTS

Various combinations of ΔX, ΔY, ΔZ, ΔT, and δ were used in the solution, all of which satisfied the stability requirements set forth by Chorin (1). It was also found, however, that unless the ratio $\Delta T/\Delta X^2$ was of the same order of magnitude as the other ratios of this type, the solution would diverge. The values of ΔX, ΔY, ΔZ, ΔT, and δ given previously were found to be the most rapidly convergent values tried. Profiles were obtained using these values and Re equal to 1.0. They are plotted in Figure 2. The development of the centerline velocity is shown in Figure 3 and Table I. Contrary to expectations, it was found that for any value of entrance length, the flow would not become fully established before it reached the X_e imposed upon it. This was true even though the X_e was much larger than the value computed by McComas (.0328). In order to test the validity of the method, a solution for fully established flow in the duct was obtained using the fully established profile as boundary conditions at X = 0 and X = X entrance length. In this case, the profile at every X-wise location should be the same. Since the X-wise station $X = X_e/2$ is the last station to be effected by the relaxation technique, the profile at $X = X_e/2$ is plotted. For 200 iterations the solution has not yet converged, and the probable error based on a calculation of $U(X_e)/U(X_e/2)$ on the centerline is 15.4 per cent. After 500 iterations, the probable error has decreased to 0.7 per cent, and after 800 iterations, the probable error is reduced to 0.1 per cent. One can see that as the number of iterations is increased, the accuracy of the

solution is increased until a very high degree of accuracy (99.93 per cent) is obtained at 1500 iterations. This is shown in Table II, Figure 4 and Figure 5.

The average friction factor obtained as a function of X in the entrance length is shown in Figure 6 and Table III. The first four to six computed points were in error due to the finite value of ΔY and ΔZ, which yield a finite value of shear stress at the entrance. In actuality the shear stress is infinite at the entrance, due to the discontinuity in the velocity at Y or $Z = 0$. In the limit as $\Delta \bar{\eta}$ approaches zero, equation (25) does, in fact, approach infinity. As the entrance length is approached, the value of $\bar{\bar{F}}$ asymptotically approaches a value of 54. This value is in very good agreement with the fully established value that was introduced into the boundary conditions. It also is in good agreement with the value suggested by McComas (4) and Lundgren (3).

V. CONCLUSIONS

As a result of this analysis, it may be concluded that:

A solution using a finite difference approximation, and relaxation type of scheme suggested by Chorin (1), leads to a convergent and accurate solution albeit with substantially more computation than is inherent to an explicit scheme. The accuracy of the solution depends on the number of iterations made, the size of the finite-difference variables ΔX, ΔY, ΔZ, and ΔT, and the value of the entrance length imposed upon the flow. This dependence of the solution upon the length of the entrance region is not what was expected and should be the subject of further investigation. The solution obtained in this analysis was based on a Reynolds number of 1.0. As one can see, the computing equations contain Reynolds number as a parameter and therefore depend upon it. Before a solution of this type may be used in the calculation of heat transfer in an entrance region, it would be necessary to investigate the behavior of this method at Reynolds numbers that are more typical of actual flow problems ($R_e = 10^3$). A further limitation to the solution obtained in this analysis is that of computer time and storage space. This might be overcome by making use of the symmetry of the duct with a suitable modification of the boundary conditions.

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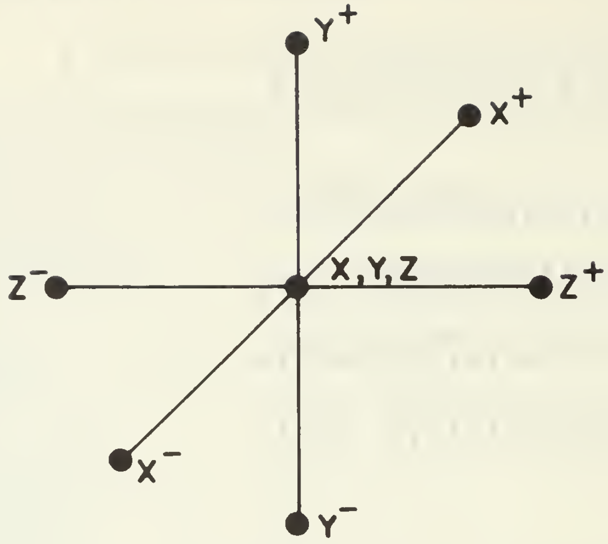
APPENDIX A

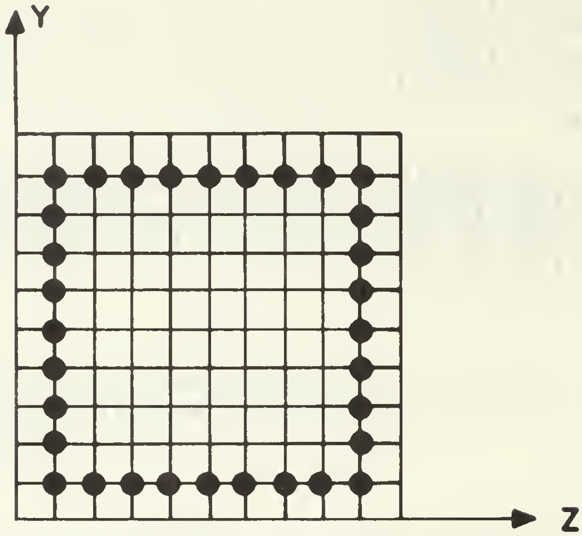
SAMPLE FORTRAN IV PROGRAM

```
C     THIS PROGRAM EVALUATES THE FULL INCOMPRESSIBLE NAVIER-STOKES
C     AND CONTINUITY EQUATIONS IN THE ENTRANCE REGION OF A SQUARE DUCT
C     YIELDING THE HYDRODYNAMIC VARIABLES U,V,W AND P AT EVERY GRID
C     POINT IN THE ENTRANCE LENGTH.
C
C
C
C
      DIMENSION U(21,11,11,3),V(21,11,11,3),W(21,11,11,3),P(21,11,11,3)
      READ(5,1)II,JJ,KK,MM,NN
    1 FORMAT(5I6)
      READ(5,2)RE,A,DT,DX,DY,DZ,DELTA
    2 FORMAT(7F10.6)
      N=1
      PI=3.1415926535897
      F=24./RE*1./(1.-192./PI**5*(TANH(PI/2.)-1./3.**5*TANH(3.*PI/2.)))
      III=II-1
      DO 3 I=1,II
      DO 3 J=1,MM
      DO 3 K=1,NN
      U(I,J,K,1)=0.0
      U(I,J,K,2)=0.0
      U(I,J,K,3)=0.0
      V(I,J,K,1)=0.0
      V(I,J,K,2)=0.0
      V(I,J,K,3)=0.0
      W(I,J,K,1)=0.0
      W(I,J,K,2)=0.0
      W(I,J,K,3)=0.0
      P(I,J,K,1)=0.0
      P(I,J,K,2)=0.0
      P(I,J,K,3)=0.0
    3 CONTINUE
      DO 11 M=1,3
      DO 11 J=2,JJ
      DO 11 K=2,KK
      Y=-.5+(J-1)/10.
      Z=-.5+(K-1)/10.
      U(1,J,K,M)=1.0
      U(II ,J,K,M)= 2./PI**3*F*RE*((1.-COSH(PI*Y)/COSH(PI/2.))*COS(PI*Z)
     *-1./27.*(1.-COSH(PI*3.*Y)/COSH(PI*3./2.))*COS(PI*3.*Z)
     *+1./125.*(1.-COSH(PI*5.*Y)/COSH(PI*5./2.))*COS(PI*5.*Z))
   11 CONTINUE
   10 CONTINUE
      DO 5 I=2,III
      DO 5 J=1,MM
      DO 5 K=1,NN
      AA=1.+2.*DT/DX**2+2.*DT/DY**2+2.*DT/DZ**2
      IF(J.EQ.1 )GO TO 13
      IF(K.EQ.1 )GO TO 13
      IF(J.EQ.11)GO TO 13
      IF(K.EQ.11)GO TO 13
C
C
      U(I,J,K,3)=(-RE*DT/DX*(U(I+1,J,K,1)**2-U(I-1,J,K,1)**2)
     *-RE*DT/DY*(U(I,J+1,K,1)*V(I,J+1,K,1)-U(I,J-1,K,1)*V(I,J-1,K,1))
     *+2.*DT/DX**2*(U(I+1,J,K,1)+U(I-1,J,K,1)-U(I,J,K,2))
     *+2.*DT/DY**2*(U(I,J+1,K,1)+U(I,J-1,K,1)-U(I,J,K,2))
     *-RE*DT/DZ*(U(I,J,K+1,1)*W(I,J,K+1,1)-U(I,J,K-1,1)*W(I,J,K-1,1))
     *+2.*DT/DZ**2*(U(I,J,K+1,1)+U(I,J,K-1,1)-U(I,J,K,2))
     *-DT/DX*1./DELTA*(P(I+1,J,K,1)-P(I-1,J,K,1))+U(I,J,K,2))*1./AA
C
C
      V(I,J,K,3)=(-RE*DT/DY*(V(I,J+1,K,1)**2-V(I,J-1,K,1)**2)
     *-RE*DT/DX*(U(I+1,J,K,1)*V(I+1,J,K,1)-U(I-1,J,K,1)*V(I-1,J,K,1))
     *-RE*DT/DZ*(W(I,J,K+1,1)*V(I,J,K+1,1)-W(I,J,K-1,1)*V(I,J,K-1,1))
     *+2.*DT/DX**2*(V(I+1,J,K,1)+V(I-1,J,K,1)-V(I,J,K,2))
     *+2.*DT/DY**2*(V(I,J+1,K,1)+V(I,J-1,K,1)-V(I,J,K,2))
     *+2.*DT/DZ**2*(V(I,J,K+1,1)+V(I,J,K-1,1)-V(I,J,K,2))
     *-DT/DY*1./DELTA*(P(I,J+1,K,1)-P(I,J-1,K,1))+V(I,J,K,2))*1./AA
C
C
      W(I,J,K,3)=(-RE*DT/DZ*(W(I,J,K+1,1)**2-W(I,J,K-1,1)**2)
```

```
     *-RE*DT/DX*(U(I+1,J,K,1)*W(I+1,J,K,1)-U(I-1,J,K,1)*W(I-1,J,K,1))
     *-RE*DT/DY*(V(I,J+1,K,1)*W(I,J+1,K,1)-V(I,J-1,K,1)*W(I,J-1,K,1))
     *+2.*DT/DX**2*(W(I+1,J,K,1)+W(I-1,J,K,1)-W(I,J,K,2))
     *+2.*DT/DY**2*(W(I,J+1,K,1)+W(I,J-1,K,1)-W(I,J,K,2))
     *+2.*DT/DZ**2*(W(I,J,K+1,1)+W(I,J,K-1,1)-W(I,J,K,2))
     *-DT/DZ*1./DELTA*(P(I,J,K+1,1)-P(I,J,K-1,1))+W(I,J,K,2))*1./AA
C
C
      P(I,J,K,3)=-DT/DX*(U(I+1,J,K,1)-U(I-1,J,K,1))
     *-DT/DY*(V(I,J+1,K,1)-V(I,J-1,K,1))
     *-DT/DZ*(W(I,J,K+1,1)-W(I,J,K-1,1))
     *+P(I,J,K,2)
   13 CONTINUE
C
      IF(K.EQ.11.AND.J.LT.11.AND.J.GT.1)
     *P(I,J,11,3)=-3.*DT/DZ*(W(I,J,10,1))
     *+P(I,J,11,2)
      IF(K.EQ.1.AND.J.GT.1.AND.J.LT.11)
     *P(I,J,1 ,3)=-3.*DT/DZ*(W(I,J,2 ,1))
     *+P(I,J,1 ,2)
      IF(J.EQ.11.AND.K.LT.11.AND.K.GT.1)
     *P(I,11,K,3)=-3.*DT/DY*(V(I,10,K,1))
     *+P(I,11,K,2)
      IF(J.EQ.1 .AND.K.GT.1.AND.K.LT.11)
     *P(I,1 ,K,3)=-3.*DT/DY*(V(I,2 ,K,1))
     *+P(I,1 ,K,2)
    5 CONTINUE
      DO 6 I=2,III
      DO 6 J=2,JJ
      DO 6 K=2,KK
      U(I,J,K,1)=U(I,J,K,3)
      V(I,J,K,1)=V(I,J,K,3)
      W(I,J,K,1)=W(I,J,K,3)
      P(I,J,K,1)=P(I,J,K,3)
      U(I,J,K,2)=U(I,J,K,1)
      V(I,J,K,2)=V(I,J,K,1)
      W(I,J,K,2)=W(I,J,K,1)
      P(I,J,K,2)=P(I,J,K,1)
    6 CONTINUE
      WRITE(6,15)N
   15 FORMAT(I5)
      IF(N.EQ.200 )GO TO 16
      IF(N.EQ.500 )GO TO 16
      IF(N.EQ.800 )GO TO 16
      IF(N.EQ.1100)GO TO 16
      IF(N.EQ.1500)GO TO 16
      N=N+1
      GO TO 10
   16 CONTINUE
   14 CONTINUE
      WRITE(6,7)
    7 FORMAT(//T3,'POSITION',T27,'U',T45,'V',T63,'W',T81,'P',T99,'T',//)
      M=3
      DO 9 I=1,II
      DO 9 J=1,MM
      DO 9 K=1,NN
      PP=P(I,J,K,M)/DELTA
      WRITE(6,8)I,J,K,U(I,J,K,M),V(I,J,K,M),W(I,J,K,M),PP,N
    8 FORMAT(3I5,4F18.8,I5)
    9 CONTINUE
  138 FORMAT(///T3,'U-CENTERLINE/U-MAX',//)
      WRITE(6,138)
      DO 137 I=1,II
      UCUM=U(I,6,6,3)/U(21,6,6,3)
      WRITE(6,139)UCUM,I
  139 FORMAT(F18.8,I6)
  137 CONTINUE
      N=N+1
      IF(N.LT.1500)GO TO 10
      RETURN
      END
```



GRID SYSTEM



GRID POINTS USED IN THE SUM FOR $\bar{F}$

FIGURE 1

GRID SYSTEM

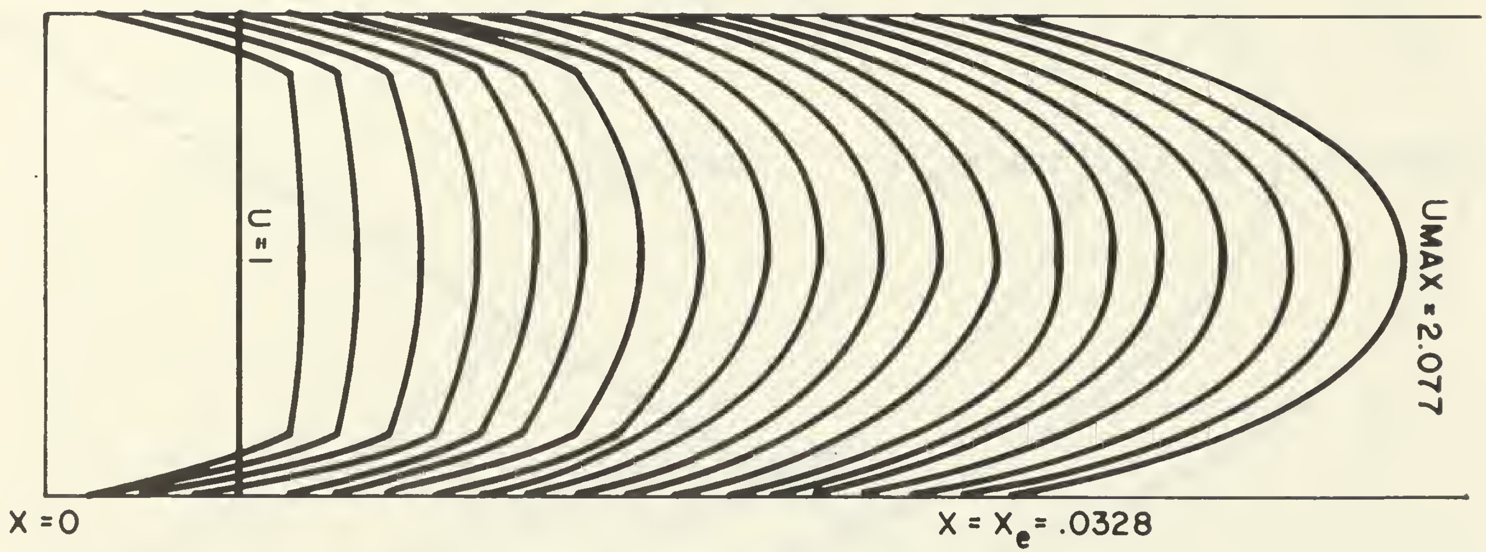


FIGURE 2
VELOCITY PROFILE DEVELOPMENT
IN
THE ENTRANCE REGION

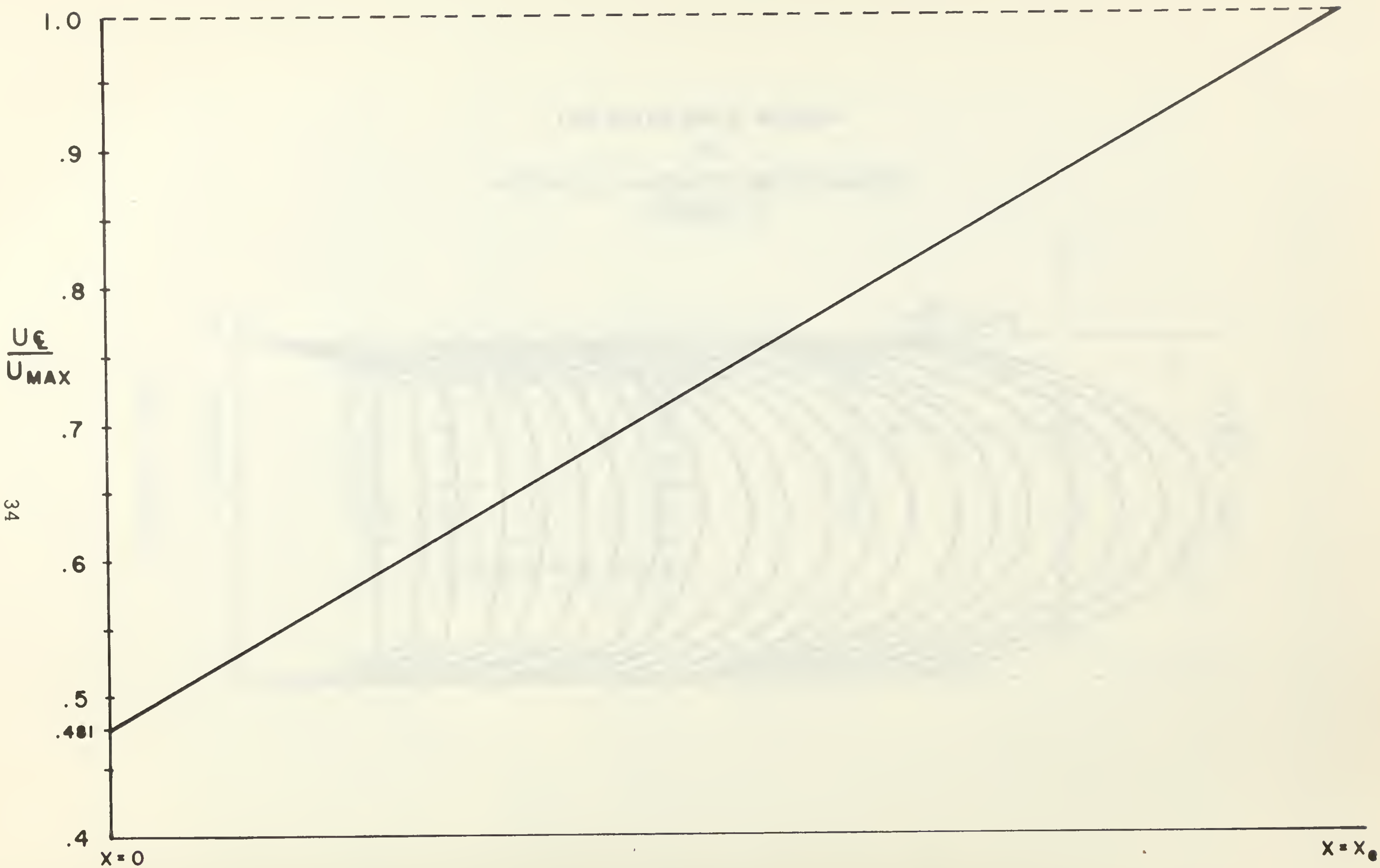


FIGURE 3

DEVELOPMENT OF THE
CENTERLINE VELOCITY

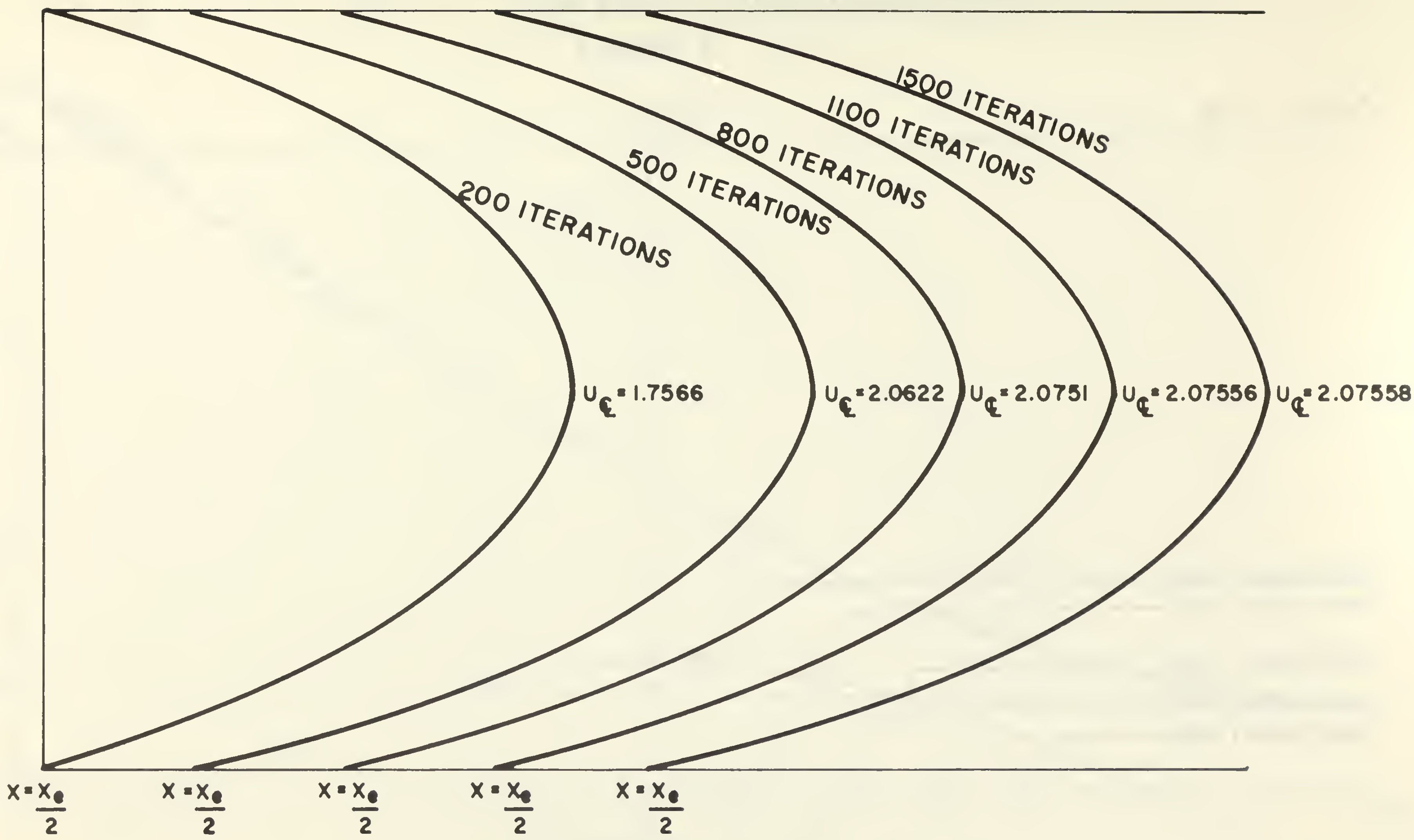


FIGURE 4

VELOCITY PROFILE DEVELOPMENT

AT $X = \frac{X_e}{2}$

FOR FULLY ESTABLISHED FLOW

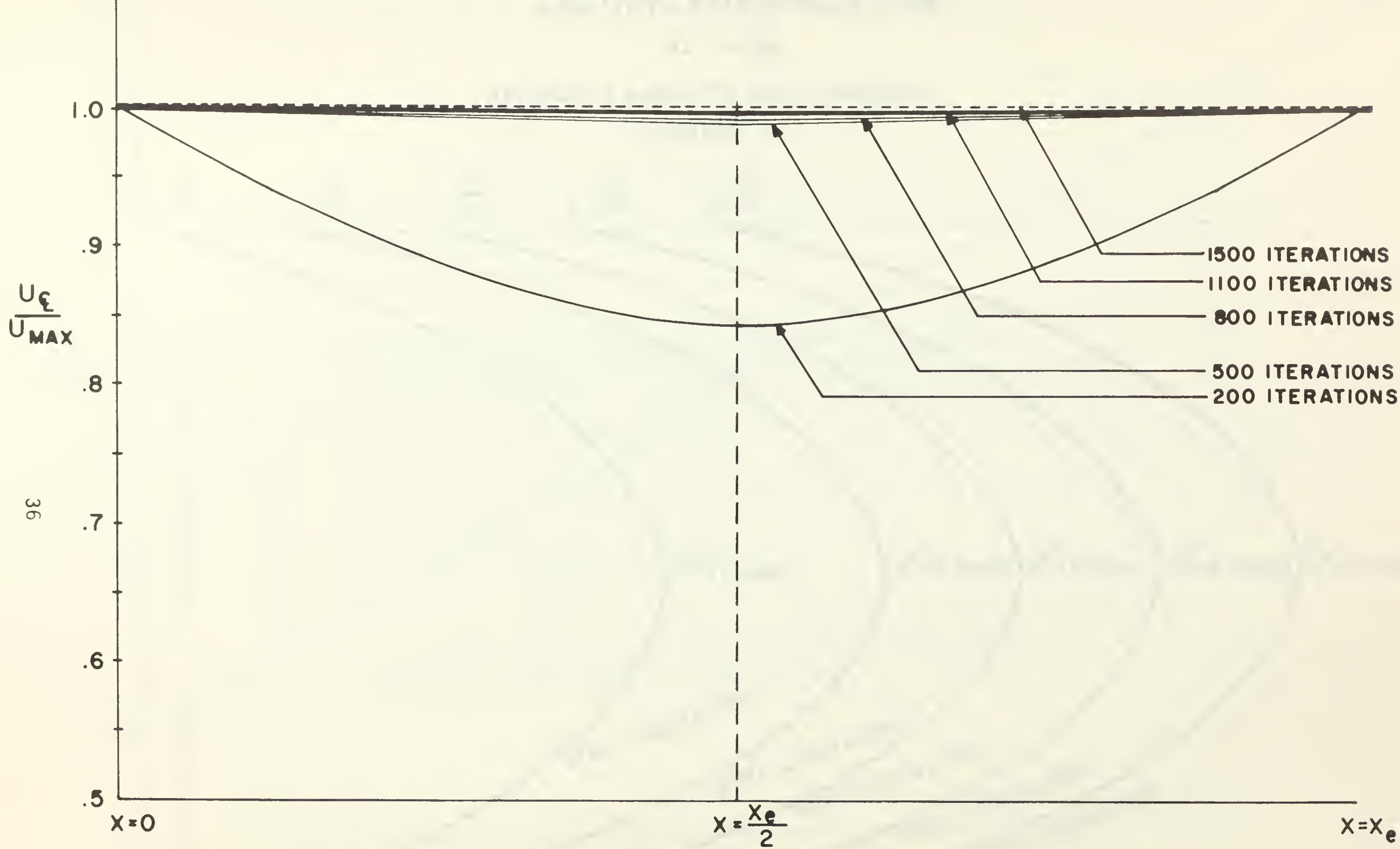


FIGURE 5

$U_{℄}/U_{MAX}$ FOR FULLY ESTABLISHED FLOW

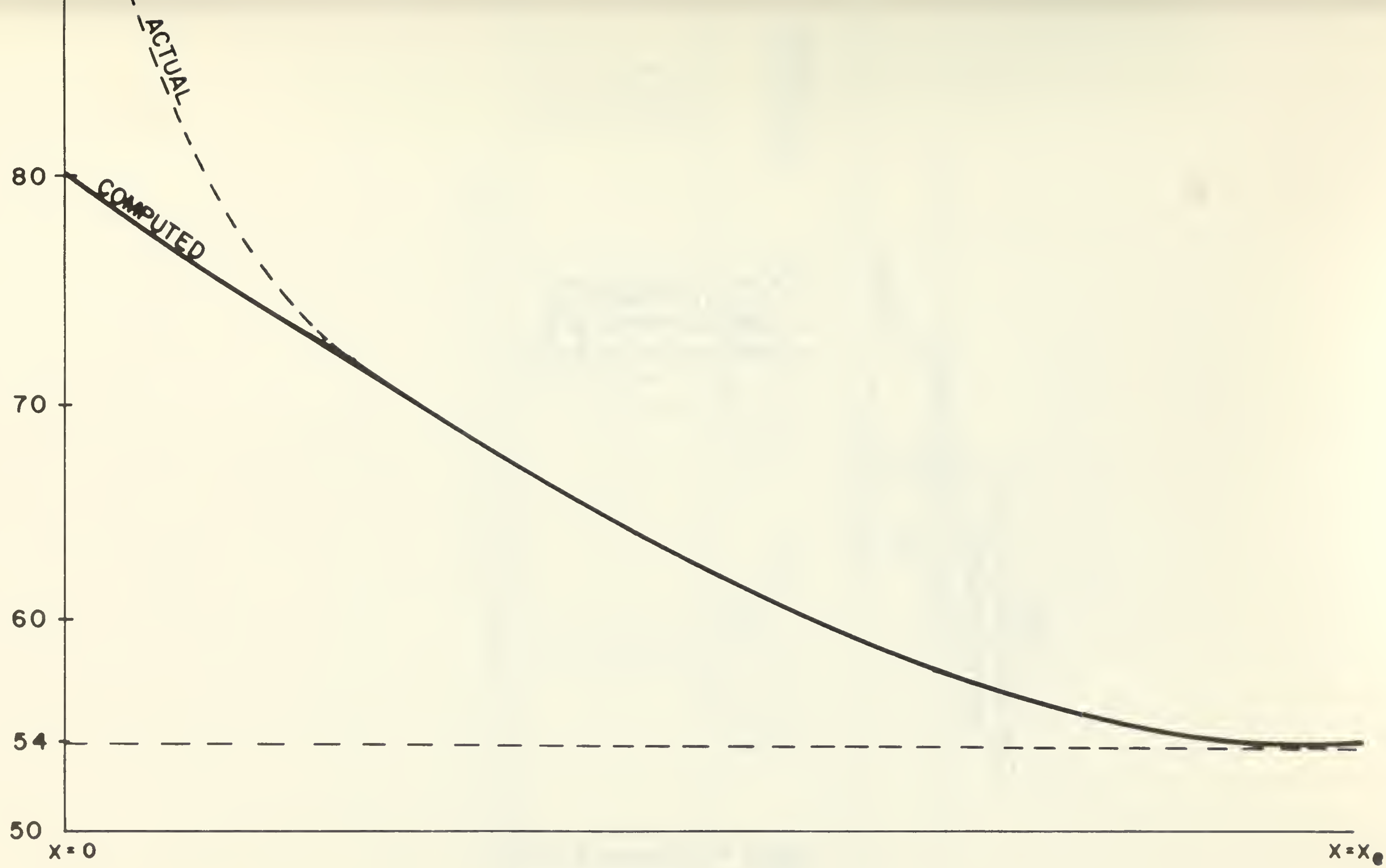


FIGURE 6
FRICTION FACTOR

TABLE I
CENTERLINE VELOCITY
DEVELOPMENT

X	$U_{℄}/U_{MAX}$
0	.4815
.00328	.5314
.00656	.5816
.00984	.6321
.01312	.6830
.01640	.7343
.01968	.7862
.02296	.8386
.02624	.8916
.02952	.9453
.03280	1.0

TABLE II
PROBABLE ERROR
OF THE
SOLUTION

NUMBER OF ITERATIONS	$U_{\text{¢}}/U_{MAX}$ AT $\frac{X=X_e}{2}$ (COMPUTED)	$U_{\text{¢}}/U_{MAX}$ AT $\frac{X=X_e}{2}$ (ACTUAL)	PROBABLE ERROR (%)
200	.84575	1.0	15.4
500	.99288	1.0	.7
800	.99908	1.0	.09
1100	.99932	1.0	.07
1500	.99933	1.0	.07

TABLE III

CALCULATED FRICTION FACTOR

X	$\bar{F}$
0	80
.00164	78.1
.00328	76.3
.00492	74.3
.00656	72.6
.00820	70.17
.00984	69.05
.01148	67.3
.01312	65.7
.01640	64.16
.01968	61.1
.02296	58.7
.02624	55.7
.02788	54.82
.02952	54.64
.03116	54.4
.03280	54.2

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13. ABSTRACT

The development of the three dimensional, laminar velocity profile in the entrance length of a rectangular duct is investigated. The solution to this hydrodynamic problem is obtained from the full, incompressible Navier-Stokes equations and the continuity equation, in finite difference form, on the digital computer employing the computational method of Chorin (1). The solution yields the hydrodynamic velocities U, V, and W and the friction factor as a function of the distance from the entrance.

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Velocity Profile Development						
Channel Flow						
Solution of Navier Stokes Equations						